The Theory of Computation
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Detailed Solution
Chapter 5, Problem 27

We need to develop a reduction from the known productive set $\overline{K}$ to the set $S(c)$ that allows us to derive the productive function for $S(c)$. Thus what we want is a reduction such that, if x is in $\overline{K}$, i.e., if $\phi_x(x)\uparrow$, then $x \in S(c)$, i.e., $\phi_x(c)\uparrow$. Define the function $\theta(x, y) = \phi_{\text{univ}}(x, x) - \phi_{\text{univ}}(x, x)$. By standard s-m-n arguments, there exists some total function f such that $\phi_{f(x)}(y) = \theta(x, y)$. But note that $\phi_{f(x)}$ is either the constant function ϵ (when $x \notin \overline{K}$), in which case $f(x) \notin S(c)$, or the totally undefined function (when $x \in \overline{K}$), in which case $f(x) \in S(c)$. Thus we have a reduction from $\overline{K}$ to $S(c)$ such that $x \in \overline{K} \Leftrightarrow f(x) \in S(c)$. (Incidentally, this reduction shows that $S(c)$ is non-r.e., not surprisingly, since every productive set is by definition non-r.e.) Now note that, if we have $dom\phi_i \subseteq S(c)$, then we have $f^{-1}(dom\phi_i) \subseteq f^{-1}(S(c)) = \overline{K}$. (We use $f^{-1}(A)$ to denote the inverse image of A under f, i.e., the set of all x such that $f(x) \in A$; we do not use f^{-1} to denote any kind of function!) Moreover, $f^{-1}(dom\phi_i)$ is r.e., so that there exists some ϕ_z such that $dom\phi_z = f^{-1}(dom\phi_i)$; to see that, just note that $\phi_z(y) = \phi_i(f(y))$ has the desired properties. But then $dom\phi_z \subseteq \overline{K}$ and so, by our previous remarks, $z \in \overline{K} - dom\phi_z$; it follows that $f(z) \in S(c) - dom\phi_i$.

So all we need is to compute the z, i.e., to find a function g such that $z = g(i)$; then the productive function for $S(c)$ is just $f \cdot g$. We saw that $\phi_z(y) = \phi_i(f(y))$ has the desired properties. Thus, if f is the (total) function ϕ_j, then we have $z = c(i, j)$ (using the composition function c) and the productive function for $S(c)$ is just $p_{S(c)}(i) = \phi_j(c(i, j))$, where j is a known constant.