The Theory of Computation
Bernard M.E. Moret
Addison-Wesley-Longman, 1998

Detailed Solution
Chapter 6, Problem 6

What we need to prove is that, if some problem A is in P and some problem B many-one reduces to A in polynomial time, then B is also in P; the we should repeat for Exp; finally we need to show that such need not be the case for E.

We begin with P. Since A is in P, there is a Turing machine M and a polynomial p such that M can decide an instance x of A in time $p(|x|)$. Let f be the polynomial-time transformation from B to A and let q be its polynomial bound. We build a decision machine M' for B by using the machine M for A and the the transformation f. Specifically, M' will decide an instance x of B by transforming it in time $q(|x|)$ to an instance $y = f(x)$ of A and running the decision procedure M on y in $p(|y|)$ time. Since the transformation takes time $q(|x|)$, what it produces, the string y, cannot exceed $q(|x|)$ in length; thus the time taken by solver M, which is $p(|y|)$, is at most $p(q(|x|))$, which is a polynomial in $|x|$. Hence our solver M' runs in time polynomial in $|x|$ and thus B is in P.

The same reasoning applies if we replace the polynomial bound p by an exponential bound of the type 2^{n^k} for some $k \in \mathbb{N}$, since we can bound any quantity of the form $2^{q^k(n)}$ by another exponential of the form 2^{n^m} for a suitable choice of m.

However, if the bound is a bound for E, that is a bound of the form 2^{kn}, then the running time after the transformation is of the form $2^{kq(n)}$, which, if q is a polynomial of degree higher than 1, cannot be bounded by any function of the form 2^{mn} for any constant m. Thus E need not be closed under polynomial-time transformations. To prove that it is actually not closed, we need a specific counterexample; that is much harder, since it requires knowledge of specific problems with specific lower bounds—we would need to know of at least one problem requiring more time than is available in E and that happens to reduce to some problem in E in polynomial time.