The Theory of Computation
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Detailed Solutions
Chapter 8, Problem 24

As in the case of *VC*, a deceptively simple iterative algorithm actually works. The algorithm is very simple indeed: start from any cut at all—even from the empty cut into $(V, \emptyset)$ if so desired. Denote the current cut by $(S, V - S)$; while moving one vertex from S to $V - S$ or from $V - S$ to S increases the number of cut edges, carry out the move and update S appropriately. Once no more improvement can be made with the transfer of a single vertex, stop and return the current cut. This is clearly a polynomial-time algorithm—even with a very na ve implementation, each step takes at most $|V|^2$ time and, since we have $|E|$ edges in all and must improve the cut by at least 1 at each step, the number of steps is bounded by $|E|$, so that even a na ve implementation must run in $O(|E| \cdot |V|^2)$ time.

We claim the the cut returned by this algorithm cuts as least half as many edges as the optimal cut. Denote an optimal cut by $(T, V - T)$; let $S_1 = S \cap T$, $S_2 = S - T$, $S_3 = T - S$, and $S_4 = V - (S \cup T)$. Then we can write the approximate solution as $(S_1 \cup S_2, S_3 \cup S_4)$ and the optimal solution as $(S_1 \cup S_3, S_2 \cup S_4)$. Let us consider what we know about the number of edges between vertices in S_i and vertices in S_j, for $1 \leq i \neq j \leq 4$, call it n_{ij}. By construction the approximate solution cannot be improved by moving any single vertex across the cut; thus, in particular, if we consider a vertex in S_1, the number of edges connecting it to other vertices in S_1 and S_2 is no larger than the number of edges connecting it to vertices in S_3 and S_4. Summing this inequality over all vertices in S_1, we get

$$2n_{11} + n_{12} \leq n_{13} + n_{14}$$

Of course, the exact same reasoning can be made about the other three subsets, so we get in all

$$\begin{cases} 2n_{11} + n_{12} \leq n_{13} + n_{14} \\ 2n_{22} + n_{12} \leq n_{23} + n_{24} \\ 2n_{33} + n_{34} \leq n_{13} + n_{23} \\ 2n_{44} + n_{34} \leq n_{14} + n_{24} \end{cases}$$

Each of these numbers is at least 0, so we can drop the first term in each inequality and write

$$\begin{cases} n_{12} \le n_{13} + n_{14} \\ n_{12} \le n_{23} + n_{24} \\ n_{34} \le n_{13} + n_{23} \\ n_{34} \le n_{14} + n_{24} \end{cases}$$

Combining the four inequalities yields

$$n_{12} + n_{34} \le n_{13} + n_{14} + n_{23} + n_{24}$$

Note that the right-hand side is exactly the number of edges cut in the approximate solution; the left-hand side is a subset of the edges cut by the optimal solution—we need to add in $n_{14} + n_{23}$. We write

$$n_{14} + n_{23} \le n_{13} + n_{14} + n_{23} + n_{24}$$

which is obviously true (again because no number is negative) and add it to our previous inequality to yield

$$n_{12} + n_{14} + n_{23} + n_{34} \le 2(n_{13} + n_{14} + n_{23} + n_{24})$$

or, in shorthand

$$\text{optimal} \le 2 \cdot \text{approximate}$$

which proves the result.