The Theory of Computation
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Solution Manual
Chapter 2, Problem 2.1

We need to show that an acyclic connected (undirected and finite) graph has exactly one fewer vertices than edges.

We proceed by induction on the number of vertices of the graph. We actually prove first that any acyclic connected graph with n vertices and at least one vertex of degree 1 has $n-1$ edges, then prove that any acyclic graph must have at least one vertex of degree less than 2.

So let us first consider connected graphs with at least one vertex of degree 1. The basis is vacuously true for a graph of $n = 1$ vertices, since such a graph cannot have any edges and is connected by default. (If you dislike starting with a basis that does not display the property very convincingly, consider a graph of 2 vertices: the two vertices must be connected by an edge for the graph to be connected and that is the only edge possible, so we have 1 edge and 2 vertices, thereby obeying the conclusion.) Assume then that all graphs on $n-1$ or fewer vertices obey the theorem and consider a connected acyclic graph $G = (V, E)$ of n vertices. If G has a vertex of degree 1, then let v be such a vertex and consider the graph G' obtained by removing from G the vertex v and its associated edge; G' is connected and has $n-1$ vertices so that it obeys the inductive hypothesis. Thus G' has $n-2$ edges and thus G must have $n-1$ edges, as desired.

What if all vertices of G have degree 2 or higher? We claim that this cannot occur, because any such graph would have a cycle. Again, we prove this by induction. The theorem is vacuously true again for graphs with one vertex or two vertices, since a degree of 2 is not possible in such graphs. (Again, if you dislike vacuous bases, consider a graph on 3 vertices where every vertex has degree 2: there is only one such graph, the complete graph on 3 vertices, K_3, and it has a cycle.) Assume then that such is true of all graphs of up to $n-1$ vertices and consider a graph G of n vertices where all vertices have degree at least 2. Let v be a vertex of lowest degree (which is at least 2) and u and w be two vertices to which v is connected. Two cases arise. If there is an edge between u and w, then G has a cycle (the three vertices u, v, and w form a cycle), as desired. If there is no edge in G between u and w, consider the new graph G' obtained from G by removing vertex v from G and adding the edge from u to w. G' has $n-1$ vertices, all of degree 2 or higher, and thus obeys the inductive hypothesis, so that it has a cycle. If this cycle does not involve the edge $\{u.w\}$, it is also present in G and we are done. If the cycle uses edge $\{u.w\}$, then G has almost the same cycle, where the edge $\{u, w\}$ (which does not exist in G) is replaced by the two edges $\{u, v\}$ and $\{v, w\}$.