The Theory of Computation

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Solution Manual

Chapter 2, Problem 2.13

- $n!$ is difficult to manipulate. The easiest way to solve this problem is to use Stirling's approximation, which, in asymptotic terms, gives $n! = \Theta(\sqrt{n}(\frac{n}{e})^n)$. Using this result, it is then easy to prove both points. n^n grows faster than $n!$, since the limit of their ratio,

$$\lim_{n\to\infty} \frac{n^n}{n!} = \lim_{n\to\infty} \frac{n^n}{\sqrt{n}(\frac{n}{e})^n} = \lim_{n\to\infty} \frac{e^n}{\sqrt{n}}$$

diverges. But $\log n^n = n \log n$, while $\log n! = \Theta(\frac{1}{2}n + n(\log n - \log e)) = \Theta(n \log n)$, so that the two functions grow at exactly the same rate. Without Stirling's approximation, we need more work, but it is also more interesting. We begin by writing the inequality that would have to be respected if n^n were $O(n!)$: for some c, N, we would have to have, for all $n > N$, $n^n \leq c \cdot n!$, or $\frac{n^n}{n!} \leq c$. Consider the ratio: if we can show that it diverges, then no suitable constant c can exist. Unfortunately, applying L'Hospital's rule right away would require differentiating $n!$. So let us instead take logs on both sides of the inequality:

$$n \log n - \sum_{i=1}^{n} \log i \leq \log c = c'$$

We can rewrite this inequality as

$$\sum_{i=1}^{n} (\log n - \log i) \leq c'$$

The sum is not easy to manipulate, but note that the first $n/2$ terms are each at least as large as $\log n - \log(n/2)$, which is in turn just 1, so that we can write (for even n)

$$\sum_{i=1}^{n} (\log n - \log i) \geq \sum_{i=1}^{n/2} (\log n - \log(n/2)) = \sum_{i=1}^{n/2} 1 = n/2$$

But that difference grows with n and thus can never be bounded by a constant c'.

For the logs, we can use ratios directly and use L'Hospital's rule to determine the convergence of these ratios. (Recall that the rule states that the limit of the ratio can be obtained by taking the limit of the ratio of the derivative of the numerator to the derivative of the denominator.) The limit of the ratio of the logarithms can be derived as follows.

$$\lim_{n\to\infty} \frac{\log n^n}{\log n!} = \lim_{n\to\infty} \frac{n \log n}{\sum_{i=1}^{n} \log i}$$

Now apply L'Hospital's rule (the derivative of $\log n$ is $1/n$ times a constant, which drops out of the ratio—we could as easily have used logarithms in base e):

$$= \lim_{n\to\infty} \frac{\log n}{\sum_{i=1}^{n} \frac{1}{i}}$$

Now note that the denominator, $\sum_{i=1}^{n} \frac{1}{i}$, is an approximation of the integral $\int_1^n \frac{1}{x} dx$; in fact, we can easily verify (e.g., visually, by looking at the rectangles given by the sum and at the curve $\frac{1}{x}$) that we have

$$\int_1^{n+1} \frac{1}{x} dx < \sum_{i=1}^{n} \frac{1}{i} < 1 + \int_1^n \frac{1}{x} dx$$

Since we have $\int_1^n \frac{1}{x} dx = \ln n$, it follows that we have $\sum_{i=1}^{n} \frac{1}{i}$ is $\Theta(\log n)$ and we are done.

- For $n^{\alpha \log n}$ to be $O(n^{\log n})$, we would have to find $c, N > 0$ such that, for all $n > N$, we have $n^{\alpha \log n} \le c \cdot (n^{\log n})$. But the limit of the ratio $\frac{n^{\alpha \log n}}{c \cdot (n^{\log n})}$ diverges, as we prove below:

$$\lim_{n\to\infty} \frac{n^{\alpha \log n}}{c \cdot (n^{\log n})} = \lim_{n\to\infty} \frac{n^{\alpha \log n - \log n}}{c} = \lim_{n\to\infty} \frac{n^{(\alpha-1)\log n}}{c}$$

Since $\alpha > 1$, the numerator is n raised to a power equal to some positive multiple of $\log n$ and thus grows as n grows, whereas the denominator is a constant; therefore, the ratio diverges in the limit and no c and N can be found that meet the requirements.

For the logs, we have $\log n^{\alpha \log n} = \alpha \log n \log n = \alpha (\log n)^2$ and $\log n^{\log n} = \log n \log n = (\log n)^2$; thus the two are within a constant ratio (α) of each other, so the result is clearly true.