The Theory of Computation
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Solution Manual
Chapter 2, Problem 2.14

Here are a few results about $O(\,)$ notation.

1. [Transitivity] If $f(n)$ is $O\big(g(n)\big)$ and $g(n)$ is $O\big(h(n)\big)$, then $f(n)$ is $O\big(h(n)\big)$.
2. [Scaling] If $f(n)$ is $O\big(g(n)\big)$, then, for any $k > 0$, $f(n)$ is $O\big(k{\cdot}g(n)\big)$.
3. [Rule of sums] If $f_1(n)$ is $O\big(g_1(n)\big)$ and $f_2(n)$ is $O\big(g_2(n)\big)$, then $\big(f_1 + f_2\big)(n)$ is $O\big(\max\big(g_1(n), g_2(n)\big)\big)$.
4. [Rule of products] If $f_1(n)$ is $O\big(g_1(n)\big)$ and $f_2(n)$ is $O\big(g_2(n)\big)$, then $(f_1 \cdot f_2)(n)$ is $O\big((g_1 \cdot g_2)(n)\big)$. Note that $(f_1 \cdot f_2)(n)$ refers to the product of the values of the two functions at n and not to the value at n of the composition of the two functions.

Proof.

1. $f(n)$ is $O\big(g(n)\big)$ says that there exist c_1 and N_1 such that $f(n) \le c_1 \cdot g(n)$ holds for all $n \ge N_1$; similarly, $g(n)$ is $O\big(h(n)\big)$ says that there exist c_2 and N_2 such that $g(n) \le c_2 \cdot h(n)$ holds for all $n \ge N_2$. Take $c = c_1 \cdot c_2$ and $N = \max(N_1, N_2)$. Now for any $n \ge n_0$, we have

$$f(n) \le c_1 \cdot g(n) \le c_1 \cdot \big(c_2 \cdot h(n)\big) = c \cdot h(n),$$

   where the first inequality follows from $n \ge N_1$ and the second from $n \ge N_2$. Thus, for any $n \ge N$, we have $f(n) \le c \cdot h(n)$, so that $f(n)$ is $O\big(h(n)\big)$.
2. Again, $f(n)$ is $O\big(g(n)\big)$ means that there exist c_1 and N_1 such that $f(n) \le c_1 \cdot g(n)$ holds for all $n \ge N_1$. Take $c = c_1/k$ and $N = N_1$. Then, for $n \ge N$, we have $f(n) \le c_1 \cdot g(n) = (ck)g(n) = c\big(k \cdot g(n)\big)$, so that $f(n)$ is $O\big(k{\cdot}g(n)\big)$.
3. Before proving this result, note that $\max\big(g_1(n), g_2(n)\big)$ is a function defined *pointwise*, i.e., independently for each n; for some n it takes on the value of g_1, and for others it takes on the value of g_2. Take $c = 2 \cdot \max(c_1, c_2)$ and $N = \max(N_1, N_2)$. The rest follows easily.

4. Here take $c = c_1 \cdot c_2$ and $N = \max(N_1, N_2)$. The rest follows easily.

Q.E.D.