The Theory of Computation
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Solution Manual
Chapter 2, Problem 2.2

We prove the the chromatic number of K_n is in two steps: first we show that n colors suffice, then we show that n colors are needed. That n colors suffice is rather trivial: since every vertex will be given its own separate color, we can never have any edge with identically colored endpoints. To verify that n colors are necessary, we use contradiction: if we use fewer than n colors, then there will be at least two identically colored vertices; but there is an edge between any two vertices in a complete graph, so that the edge between these two vertices will violate the requirements of a legal coloring.

To see that a tree with at least two vertices always has chromatic number 2, consider the simple coloring strategy: color the root red; then color the children black; then color the grandchildren red; and so on, alternating colors as we alternate odd and even levels. The result is a legal coloring because the only edges present in the tree are between adjacent levels—which always have differently colored vertices. By our previous result, two colors are needed as soon as the graph includes a K_2—which is just a single edge. Any tree with two or more vertices has at least one edge and thus requires two colors.