The Theory of Computation

Bernard M.E. Moret

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Solution Manual
Chapter 2, Problem 2.24

We can combine our result of Exercise 2.16—that the sum of the degrees of the vertices of undirected graph (or the sum of the outdegrees and indegrees of the vertices of a directed graph) is exactly twice the number of edges (or arcs) of the graph—with that of Exercise 2.1—that a tree on n vertices has $n-1$ edges—to obtain directly our first result: the sum of the degree of the vertices of a tree of n vertices is $2(n-1)$.

The converse is harder. We prove it by induction on n, the number of nodes. The basis (for $n=1$) is vacuously true, since the degree is 0 and since there exists a single tree with one vertex and no edge. Assume then that the result holds for all values of i up to $n-1$ and consider a collection of n natural numbers, $\{d_1, d_2, \ldots, d_n\}$s (where we have $n > 1$) obeying $\sum_{i=1}^{n} d_i = 2(n-1)$. At least one of the d_i must equal 1, since otherwise (all d_i are at least 2), we would have a sum at least as large as $2n$. Let j be the index of one such value, i.e., we have $d_j = 1$. Thus vertex j is a leaf of the putative tree we want to build; but note that a leaf can be attached to *any* non-leaf node of the tree—that is, it has no "structural" effect on the tree. Thus we remove d_j from our collection and decrease one of the other values by 1, making sure to pick one that is at least 2 (i.e., that is not another leaf). (Again, there must be at least one d_i with a value of at least 2, since otherwise the sum would only equal n.) Now the resulting collection has $n-1$ numbers and its sum is the old sum, $2(n-1)$, minus 2 (the removed d_j and the one decremented value), that is, its sum is $2(n-1)-2 = 2(n-2)$. Thus the resulting collection obeys the inductive hypothesis and can indeed be mapped into a tree; once that is done, we attach to (any node of) this tree one more leaf and we have our desired tree.