The Theory of Computation
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Solution Manual
Chapter 2, Problem 2.38

Actually, it is fairly easy to design new pairing functions: all we need to do is to picture sweeping the upper quadrant in some systematic way. The interesting parts come in making the function monotone in both arguments and in devising a way to compute it and invert it.

Here is a fairly simple example, which is really a variation on the basic dovetailing function defined in the text. Imagine sweeping the upper quadrant by running right along a horizontal sweep up to a point, then running up along a vertical sweep at a coordinate just one larger than that reached in the horizontal sweep. Formally, we sweep in order the pairs

$$(i, 1), (i, 2), \ldots, (i, i), (i+1, 1), (i+1, 2), \ldots, (i+1, i),$$

then start again with the pairs

$$(i+1, 1), (i+1, 2), \ldots, (i+1, i+1), (i+2, 1), (i+2, 2), \ldots, \\ (i+2, i+1),$$

and so on, starting with the sequence

$$(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (2, 1), \ldots$$

It is clear that this sequence goes through all pairs; it is also clear that it is monotone in both arguments. Computing it is not too hard, nor is inverting it: the key is to note that the switching point from a horizontal sweep to a vertical one occurs on the diagonal and so after enumerating i^2 pairs. Thus we can write $\langle i, i\rangle = i^2$ and hence also $\langle i-k, i\rangle = i^2 - k$, for $0 \le k \le i$; finally, we can write $\langle i+1, k\rangle = i^2 + k$ for $0 \le k \le i$. Inverting the pairing thus also relies on the switching point and starts by computing $n - \lfloor\sqrt{n}\rfloor^2$ and checking it against $\lfloor\sqrt{n}\rfloor$.