The Theory of Computation

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Solution Manual
Chapter 2, Problem 2.44

The hint pretty much solves the problem. Basically, we dovetail on n, the degree of the polynomial, and on $n+1$ copies of the natural numbers to assign the $n+1$ coefficients.

Formally, we proceed by induction. For $n=0$, the set of polynomials of degree 0 is just $\mathbb{Z}$, which is known to be countable. Assume that the set of all polynomials of degree not exceeding $n-1$ is countable and consider the set of all polynomials of degree not exceeding n. This set consists of all polynomials of degree not exceeding $n-1$, a countable set by inductive hypothesis, and of all polynomials of degree n. Thus, by Exercise 2.42, we need only show that the set of polynomials of degree n is countable, since the union of two countable sets is itself countable. Any polynomial of degree n can be written as $a_n x^n + p_{n-1}(x)$, where $p_{n-1}(x)$ is a polynomial of degree not exceeding $n-1$. by inductive hypothesis, we have a procedure to enumerate the latter type of polynomial; the first term is simply a matter of choosing $a_n \in \mathbb{Z}$. Thus we can dovetail on the value of a_n (all integers) and on all polynomials of degree not exceeding $n-1$ and enumerate all polynomials of degree n.