The Theory of Computation

Bernard M.E. Moret

Addison-Wesley-Longman, 1998

Solution Manual

Chapter 3, Problem 3.16

1. $(P^*)^* = P^*$
   True. We prove it by induction. Our base case is to show that $P^* P^* = P^*$; from this base case, the conclusion follows by a trivial induction on the definition of Kleene closure: $(P^*)^* = \bigcup_{i \in \mathfrak{N}} (P^*)^i$ and our induction gives us that $(P^*)^i = P^*$. To prove the base case, we use a separate, nested induction: since $P^* = \bigcup_{i \in \mathfrak{N}} P^i$ and since $P^i P^j = P^{i+j}$, we get

$$P^* P^* = \bigcup_{i \in \mathfrak{N}} P^i \bigcup_{j \in \mathfrak{N}} P^j = \bigcup_{i,j \in \mathfrak{N}} P^{i+j} = \bigcup_{i \in \mathfrak{N}} P^i = P^*$$

2. $(P + Q)^* = (P^* Q^*)^*$
   True. We prove it by showing inclusion in both directions. To show that $A = (P + Q)^*$ is contained in $B = (P^* Q^*)^*$, we prove by induction that $(P + Q)^i$ is contained in B for each i. Clearly $(P + Q)^0 = \epsilon$ and $(P + Q)$ are both contained in B. If $(P + Q)^i$ is contained in B, then $(P + Q)^{i+1} = (P + Q)^i (P + Q)$ contains concatenations of strings from $(P + Q)^i$, which is contained in B, with strings from $(P + Q)$, which is also contained in B; since B is a Kleene closure, it contains all concatenations of any of its strings with any other of its strings, and so contains $(P + Q)^i (P + Q)$. To show that B is contained in A, we use induction on the definition of Kleene closure and prove that $(P^* Q^*)^i$ is contained in A for each i. Clearly, $(P^* Q^*)^0 = \epsilon$ and $(P^* Q^*)$ are contained in A (the latter would, in all formality, require another nested induction). If $(P^* Q^*)^i$ is contained in A, then $(P^* Q^*)^{i+1} = (P^* Q^*)^i (P^* Q^*)$ contains concatenations of strings from $(P^* Q^*)^i$, which is contained in A, with strings from $(P^* Q^*)$, which is aslo contained in A; since A is a Kleene closure, it contains all concatenations of any of its strings with any other of its strings, and so contains $(P^* Q^*)^i (P^* Q^*)$.

3. $(P + Q)^* = P^* + Q^*$
False. An easy counterexample is $(0 + 1)^*$, which is Σ^* on the alphabet $\Sigma = \{0, 1\}$; the regular expression $0^* + 1^*$ denotes only strings of 0s and strings of 1s, not strings containing both 0s and 1s.