The Theory of Computation

Bernard M.E. Moret

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Solution Manual
Chapter 5, Problem 5.18

If the set S is empty, the results are trivially true; so let us assume that the set S is non-empty and let f be some recursive function whose range is the set S (one such function must exist by definition of r.e. sets).

We begin with the ranges. and define the new function

$$\theta(z) = \phi_{\text{univ}}(f(\Pi_1(z)), \Pi_2(z))$$

That is, $\theta(\langle x, y\rangle)$ runs the xth program in S (namely $\phi_{f(x)}$) on argument y; the result is, by definition, in the range of $\phi_{f(x)}$ and thus in R. Thus $ran\theta \subseteq R$. Conversely, note that any element $x \in R$ is in $ran\phi_i$ for some $i \in S$; in particular, there exists some y such that $\phi_i(y) = x$. Since f enumerates S, there exists some j such that $f(j) = i$; but then $\theta(\langle j, y\rangle) = \phi_{f(j)}(y) = \phi_i(y) = x$, so that $x \in ran\theta$ and thus $R \subseteq ran\theta$.

Now we move to the domains. What we need to do here is to define a new function ζ that will be defined on any value on which one of the ϕ_x, $x \in S$, is defined. We cannot afford to run the ϕ_x on a given value directly, because it might be undefined and cause our ζ to become undefined; thus we use our old trick of dovetailing with a step function:

$$\zeta(x) = \mu z[\text{step}(f(\Pi_1(z)), x, \Pi_2(z)) > 0]$$

If $x \in dom\zeta$, then there exists $z = \langle i, j\rangle$ such that $\text{step}(f(i), x, j) > 0$, i.e., such that $\phi_{f(i)}(x)\downarrow$, i.e., such that $x \in dom\phi_{f(i)}$, i.e., such that $x \in D$; hence $dom\zeta \subseteq D$. Conversely, if $x \in D$, then there exists $k \in S$ such that $x \in dom\phi_k$; hence there must exist some number of steps, say t, sufficient to ensure that ϕ_k converges on x, i.e., such that $\text{step}(k, x, t) > 0$. But f enumerates S, so that there must exist y such that $f(y) = k$; but then we have $\zeta(x) \le \langle y, t\rangle$, so that $\zeta(x)\downarrow$, so that $x \in dom\zeta$; hence $D \subseteq dom\zeta$, as desired.