The Theory of Computation
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Solution Manual
Chapter 5, Problem 5.19

- That is clear from the definition. To be formal, we define the characteristic function of K_t:

$$c_{K_t}(x) = \begin{cases} 1 & \exists y \leq t \, [\text{step}(x, x, y) > 0] \\ 0 & \text{otherwise} \end{cases}$$

  (or just $c_{K_t}(x) = \text{lev}(\text{step}(x, x, t))$, under the usual assumption that, once a function has converged after some n_0 steps, it also converges for all $n \geq n_0$ steps). Since the predicate is recursive (it is even primitive recursive if we have a primitive recursive step function), we obtain a valid recursive characteristic function. The set K_t is a "sanitized," step-bounded version of K. In particular, note that $K_t \subseteq K_{t+1} \subset K$, with $\lim_{t \longrightarrow \infty} K_t = K$, so that

$$\bigcup_{t \in \mathbb{N}} K_t = K$$

  (which incidentally fits the result of Problem 2, since all sets involved are r.e.).
- We use the family of K_t sets to provide a counterexample. Note that $\overline{K} \subset \overline{K}_{t+1} \subseteq \overline{K}_t$, with $\lim_{t \longrightarrow \infty} \overline{K}_t = \overline{K}$, so that

$$\bigcap_{t \in \mathbb{N}} \overline{K_t} = \overline{K}$$

  Since $\mathbb{N}$ and the K_ts are recursive and thus r.e., this is an example of an intersection of r.e. sets indexed by an r.e. set that does not produce an r.e. set. However, the intersection is not precisely of the form given in our problem, since $\overline{K_t}$ is unlikely to equal $dom \phi_t$. We can make the idea work, however, by noting that each $\overline{K_t}$ is recursive, and thus r.e., and thus the domain of some partial recursive function, call it $\phi_{g(t)}$.

Now let $S = g(\mathbb{N})$; S is clearly an r.e. set if g is a recursive function. Then

$$\bigcap_{x \in S} dom\phi_x = \bigcap_{t \in \mathbb{N}} dom\phi_{g(t)} = \bigcap_{t \in \mathbb{N}} \overline{K_t} = \overline{K}$$

and is not r.e. Thus it remains only to show that g is a total recursive function; define

$$\theta(t, x) = \begin{cases} 1 & \exists y \leq t \, [\text{step}(x, x, y) > 0] \\ \text{undefined} & \text{otherwise} \end{cases}$$

Use the s-m-n theorem to obtain $\phi_{s(t)}(x) = \theta(t, x)$. Note that $dom\phi_{s(t)} = K_t$ by definition; hence $s(t)$ is our desired g.