The Theory of Computation

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Solution Manual
Chapter 5, Problem 5.2

We follow the text in defining the function $p(n, x)$, which is intended to return the value $\langle n + 1, f(n, x), \ldots, f(), x)\rangle_{n+2}$, using primitive recursion:

$$p(0, x) = \langle 1, g(x)\rangle$$

which is a primitive recursive function of one variable, as desired, and

$$p(n + 1, x) = q(n, p(n, x), x)$$

where q is defined through substitution as

$$q(a, b, c) = \langle \text{Succ}(\text{Succ}(\Pi_1^3(a, b, c))), \rangle h(\Pi_1^3(a, b, c), \Pi_3^3(a, b, c), \Pi_2^3(a, b, c)), \Pi_2(\Pi_2^3(a, b, c))\rangle\rangle$$

where each function used is primitive recursive (but where we have slightly simplified the formal definition of q through substitution in order to avoid a long list of intermediate functions).

Now we can formally define f through primitive recursion as

$$\begin{cases} f(0, x) = g(x) \\ f(n + 1, x) = h(\Pi_1^3(n, f(n, x), x), \Pi_3^3(n, f(n, x), x), p(\Pi_1^3(n, f(n, x), x), \Pi_3^3(n, f(n, x), x))) \end{cases}$$

where no recursion is used directly—the only use of the primitive recursive mechanism is in distinguishing the case $n = 0$ from the others.