# The Theory of Computation

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## *Solution Manual*
## *Chapter 5, Problem 5.22*

$S$ is neither recursive nor r.e. That it is not recursive is an immediate consequence of Rice's theorem. Alternatively, we can reduce $K$ to $S$ as follows. Let $i$ be the index of some not totally undefined function and define $\theta(x, y) = \text{Zero}(\phi_{\text{univ}}(x, x)) + \phi_i(y)$. Since $\theta$ is partial recursive, there exists some index $k$ such that $\phi_k(x, y) = \theta(x, y)$ and, by the s-1-1 construction, there exists some index $l = s(k, x)$ such that $\phi_l(y) = \phi_k(x, y)$ for each fixed $x$. Now note that

$$\phi_l(y) = \phi_k(x, y) = \theta(x, y) = \text{Zero}(\phi_{\text{univ}}(x, x)) + \phi_i(y)$$

is totally undefined if $x \notin K$ and is the function $\phi_i$ otherwise. Thus we have $\{l, i\} \in S$ if and only if $x \in K$; that is, we can decide whether or not $x \in K$ by deciding whether or not $h(x) = \{s(k, x), i\} \in S$, where $i$ and $k$ are known constants. Since $K$ is not recursive, neither is $S$.

Now, it would seem that the complement of $S$ is r.e.: we can run, using the step function, a dovetailing computation on $\phi_i$ and $\phi_j$ to check, for each possible argument $x$, whether $\phi_i(x)$ and $\phi_j(x)$ are both defined and different, in which case we can immediately answer yes (i.e., print the pair $\{i, j\}$ as belonging to $\overline{S}$). The problem with that argument is that it ignores pairs $\{i, j\}$ that belong to $\overline{S}$ because one function is defined where the other is not (suppose, for instance, that $\phi_i$ is the totally undefined function), a difference that cannot be affirmed through enumeration. So, we should in fact conclude that neither $S$ nor its complement is r.e., which we can prove by reductions from the set $\overline{K}$.

To prove that $S$ is not r.e., we use exactly the same construction as in the previous reduction; now let $j_0$ be the index of the totally undefined function. Then we have $\{l, j_0\} \in S$ if and only if $x \notin K$ if and only if $x \in \overline{K}$. Hence membership in $S$ and membership in $\overline{K}$ are equally hard to determine, so that $S$ is not r.e. (If we could semi-decide $S$, we could also semi-decide $\overline{K}$ by computing $l = s(k, x)$ from $x$, then asking whether $\{l, j_0\}$ is in $S$; since we cannot semi-decide $\overline{K}$, neither can we semi-decide $S$.)

Proving that $\overline{S}$ is not r.e. was not asked. We can do it by the same reduction from $\overline{K}$. Remember that the resulting function, $\phi_{s(k,x)}$, is totally undefined if $x \in \overline{K}$ and is the not totally undefined function $\phi_i$ otherwise. Now the pair $\{i, s(j, x)\}$ is in $\overline{S}$ (i.e., $\phi_{s(j,x)}$ does not compute the same function as $\phi_i$) if and only if $x \in \overline{K}$. Since $\overline{K}$ is not r.e., neither is $\overline{S}$.