The Theory of Computation
Bernard M.E. Moret
Addison-Wesley-Longman, 1998

Solution Manual
Chapter 5, Problem 5.23

Both sets are non-recursive, although we cannot use Rice's theorem, since the property defining the sets is index-dependent (it is a property of programs but not of functions). So we prove that they are non-recursive by direct reduction from K. Define the function $\theta(x, y) = \text{Zero}(\phi_{\text{univ}}(x, x))$ (that will be for A; for B, we will use $\text{Succ}(\theta(x, y))$. This is clearly a partial recursive function and as such has an index, say j; we can then use the s-m-n theorem to obtain the new function $\phi_{s(j,x)}(y) = \theta(x, y)$. Now, if $x \in K$, then $\phi_{s(j,x)}(y)$ is the constant function 0 (or the constant function 1 if we take the successor), yet, if $x \notin K$, then it is the totally undefined function. Thus, if $x \notin K$, then $s(j, x)$ belongs neither to A nor to B, whereas, if $x \in K$, then $s(j, x)$ belongs to A (or to B if we used the successor). This establishes a reduction from K to A (resp., B) and thus proves that neither A nor B is recursive. On the other hand, both sets are r.e.: we can run a dovetailing process on all indices and all steps; whenever we find a z such that $\text{step}(\Pi_1(z), \Pi_1(z), \Pi_2(z)) = 1$ (1 for the set A, but 2 for the set B), we print $\Pi_1(z)$.

Now assume that there exists some recursive set C that separates A and B, i.e., such that $A \subseteq C$ and $B \subseteq \overline{C}$, and let ϕ_j (a total function) be its characteristic function. What is the value of $\phi_j(j)$? Since ϕ_j is a characteristic function, the value must be either 0 or 1. If the value is 0, this means that j is not in C; but note that we then also have $j \in A$ by definition of A; yet $A \subseteq C$, so we have a contradiction: j is in A which is inside C, yet j is not in C. Conversely, if the value is 1, then this means that j is in C; but we then also have $j \in B$ by definition of B; yet B is disjoint from C, so j cannot be both in C and in B, another contradiction. Hence there is no value that $\phi_j(j)$ can assume without causing a contradiction, so that ϕ_j cannot exist (at least not as a total recursive function) and thus C cannot be a recursive set.