The Theory of Computation
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Solution Manual
Chapter 5, Problem 5.24

1. Note that the subscript i of ψ_i is *not* an index in a programming system; it is instead used simply to identify a certain function in a convenient manner. Indeed, it is to be viewed as the result of a pairing; we can write it $i = \langle n_i, i_0, i_1, \ldots, i_{n_i} \rangle$ or, when we just want to look at two pieces, $i = \langle n_i, w_i \rangle$. (We should view i as the pairing of a count n_i and of n_i items following the count, because of the specific use of the projection functions in the definition of f.) With this notation in mind, we see that $\psi_i(x)$ is defined whenever we have $x \leq n_i$ and undefined otherwise; when it is defined, say when $x = k$ for some k, $0 \leq k \leq n_i$, then its value is i_k. Now let g be some arbitrary function from $\{0, 1, 2, \ldots, n\}$ to $\mathbb{N}$; consider the pairing

$$z = \langle n, g(0), g(1), g(2), \ldots, g(n) \rangle_{n+2}$$

   It is clear that ψ_z is the function g, so that any arbitrary function from a finite initial subset of $\mathbb{N}$ to $\mathbb{N}$ has a corresponding ψ_i.
   Since pairing functions are injective, $i \neq j$ implies $n_i \neq n_j$ or $w_i \neq w_j$. If $n_i \neq n_j$, then ψ_i and ψ_j are not defined on the same interval and thus are distinct; if $n_i = n_j$, then $w_i \neq w_j$ and thus there exists some index k, $0 \leq k \leq n_i$, such that $i_k \neq j_k$; but then $\psi_i(k) = i_k \neq j_k = \psi_j(k)$ and thus the two functions are distinct.
   Hence each ψ_i corresponds to its own specific function from an initial subset of $\mathbb{N}$ to $\mathbb{N}$.
2. It is rather interesting that such a small change in the definition would generalize from initial subsets to any finite subsets. In order to encode a mapping ϕ from any finite subset of $\mathbb{N}$ to $\mathbb{N}$, we need to use either a sparse encoding, where only the defined values are encoded (but then they must be encoded as pairs, with the value on which the function is defined paired up with the value of the function—i.e., a pair of the form $\langle k, \phi(k) \rangle$), or a dense encoding, where the value of

the function on every argument from 0 to the largest value on which the function is defined is represented. In the latter case, we must then adopt some convention for representing undefined function values. We have already seen one such convention with the step function: we represent undefined by 0 and add 1 to all defined values. This is the encoding adopted by the family $\{\pi_i\}$: note that they decrement the value returned by f through unpairing and are undefined whenever that value is 0.

More concretely, them we should think of the subscript i as a pairing of the form $i = \langle n_i, i_0, i_1, \ldots, i_{n_i} \rangle$ and of the values i_k as either 0 when the function is undefined on argument k or one more than the value of the function at k. In that light, n_i is simply the largest argument on which the function is defined.

Once this structure is understood, everything else follows easily as above. Once again, each π_i corresponds to its own specific function from a finite subset of $\mathbb{N}$ to $\mathbb{N}$.