The Theory of Computation

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Solution Manual
Chapter 5, Problem 5.25

This is the proof of the Rice-Shapiro theorem. Since the theorem offers an exact characterization, our proof has two parts: we need to show that any set with the given characteristics is r.e. and, separately, that any r.e. sets has these characteristics.

Let us then assume that our set $S = \{x \mid \phi_x \in \mathcal{C}\}$ is such that there exists an r.e. set I such that $\phi_x \in \mathcal{C}$ holds if and only if there exists some $i \in I$ with $\pi_i \subseteq \phi_x$. Note that, given x and i, we can semi-decide $\pi \subseteq \phi_x$, because there are only finitely many arguments on which π_i is defined and we can run a dovetailing computation of ϕ_x on these arguments. If we get results that differ from π_i, we can answer no; if we get convergence and agreement on all values, we can answer yes; it is only in the case where ϕ_x diverges on one of the arguments on which π_i is defined (a case in which the answer should be no) that we are unable to decide. Thus we can decide all the "yes" instances—i.e., we can semi-decide. Now, the set I is itself r.e., so we can add one level of dovetailing: given some input x, we dovetail on all $i \in I$, all arguments on which π_i converges, and all steps, and accept x whenever we have found convergence and agreement for some π_i. Hence our set S is r.e.

For the converse, let us assume that the set $S = \{x \mid \phi_x \in \mathcal{C}\}$ is r.e. We want to define an r.e. set I with the required property; in other words, we want to enumerate a collection of π_i with the required property. We can use an s-1-1 construction to obtain a total function f with $\phi_{f(i)} = \pi_i$ for all $i \in \mathbb{N}$. Let ϕ be a domain function for S, i.e., such that $dom\phi = S$. We simply define $I = dom(\phi \cdot f)$. By definition, I is r.e.; by construction, we have $i \in I \Leftrightarrow \pi_i \in \mathcal{C}$. It remains that show that this property extends from the finite behavior π_i to the full function $\phi_j \in \mathcal{C}$.

We claim that we have

$$\phi_j \in \mathcal{C} \Leftrightarrow \exists i \in I,\ \pi_i \subseteq \phi_j$$

The (⇒) part is fairly simple: if ϕ_j is in $\mathscr{C}$, define the new function

$$\theta(x, y) = \begin{cases} \phi_j(y) & \forall z \leq y\ [\text{step}(x, x, z) = 0] \\ \text{undefined} & \text{otherwise} \end{cases}$$

Using an s-1-1 construction, we write $\phi_{g(x)}(y) = \theta(x, y)$. Note that, if x is in K, then $\phi_{g(x)}$ is a finite subfunction of ϕ_j, whereas, if x is not in K, then $\phi_{g(x)}$ equals ϕ_j. Now, if ϕ_j had no finite subfunction π_i in $\mathscr{C}$, then g would be a many-one reduction from $\overline{K}$ to S, contradicting the fact that S is r.e. (If x is in $\overline{K}$, then $\phi_{g(x)}$ equals ϕ_j, which is in S, but otherwise $\phi_{g(x)}$ has finite domain and thus cannot be in S.) Thus ϕ_j must have some finite subfunction π_i in $\mathscr{C}$, as desired.

The reverse direction is more subtle. We need to show that any partial recursive function that extends a function in $\mathscr{C}$ must also be in $\mathscr{C}$—perhaps the most fundamental property of r.e. sets! This property is true whether or not the function to be extended has finite domain. So let ϕ_j be some function in $\mathscr{C}$ and let ψ be some partial recursive function with $\phi_j \subseteq \psi$; we want to prove $\psi \in \mathscr{C}$. Define

$$\theta(x, y) = \begin{cases} \psi(y) & x \in K \\ \phi_j(y) & x \notin K \end{cases}$$

(We shall take up shortly how such a definition can be implemented within the partial recursive functions.) Using an s-1-1 construction, we get $\phi_{g(x)}(y) = \theta(x, y)$. Now, if ψ were not in $\mathscr{C}$, g would be a many-one reduction from $\overline{K}$ to S, contradicting the fact that S is r.e. (If x is in $\overline{K}$, then $\phi_{g(x)}$ equals ϕ_j, which is in S, but otherwise $\phi_{g(x)}$ equals ψ which is not in S.) Thus ψ must be a function in $\mathscr{C}$, as desired.

To finish our proof, we need to show that the definition of θ just given is partial recursive. Basically, the idea is to dovetail a computation of $\phi_j(y)$ and an enumeration of K. If $\phi_j(y)$ converges before x appears in the enumeration of K, we take advantage of the fact that ψ agrees with ϕ_j everywhere the latter is defined and return $\phi_j(y)$, which is also $\psi(y)$—although we have not determined which case applies, it does not matter. If x turns up in the enumeration of K before $\phi_j(y)$ converges, then we stop computing $\phi_j(y)$ and start computing instead $\psi(y)$ and let that computation define $\theta(x, y)$. Specifying the step functions and the dovetailing formally is a big mess, so we leave it at this intuitive level.