The Theory of Computation

Bernard M.E. Moret

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Solution Manual

Chapter 5, Problem 5.26

1. Yes, such an index n exists. Define the function

$$\theta(x, y) = \begin{cases} 0 & y = x^2 \\ \text{undefined} & \text{otherwise} \end{cases}$$

   Using an s-1-1 construction, there exists a total function f such that $\phi_{f(x)}(y) = \theta(x, y)$. By the recursion theorem, f has a fixed point n such that $\phi_n = \phi_{f(n)}$. But then we have $\phi_n(y) = \phi_{f(n)}(y) = \theta(n, y)$ and thus $\phi_n(y)$ is defined only $y = n^2$. Thus the index n is the desired answer.
2. Yes, such an index n exists. Define the function

$$\theta(x, y) = \begin{cases} 0 & x \neq y \\ \text{undefined} & x = y \end{cases}$$

   Using an s-1-1 construction, there exists a total function f such that $\phi_{f(x)}(y) = \theta(x, y)$. By the recursion theorem, f has a fixed point n such that $\phi_n = \phi_{f(n)}$. But then we have $\phi_n(y) = \phi_{f(n)}(y) = \theta(n, y)$ and thus $\phi_n(y)$ is defined everywhere except for $y = n$—in other words, its domain is $\mathbb{N} - \{n\}$. Thus the index n is the desired answer.
3. Yes, such an index n exists. Define the function

$$\theta(x, y) = \begin{cases} 0 & y = x \\ \phi_{\text{univ}}(y, y) & \text{otherwise} \end{cases}$$

   Using an s-1-1 construction, there exists a total function f such that $\phi_{f(x)}(y) = \theta(x, y)$. By the recursion theorem, f has a fixed point n such that $\phi_n = \phi_{f(n)}$. But then we have $\phi_n(y) = \phi_{f(n)}(y) = \theta(n, y)$ and thus $\phi_n(y)$ is defined for $y = n$ as well as for any value of y in K and undefined everywhere else. other words, its domain is $K \cup \{n\}$. Thus the index n is the desired answer.