The Theory of Computation
Bernard M.E. Moret
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Solution Manual
Chapter 5, Problem 5.28

We are going to use the productive function f to enumerate an infinite subset of the productive set A. We proceed inductively. We can start with the totally undefined function; let i be the index of a program for that function. Since the empty set is a subset of A, we have $dom\phi_i\emptyset \subseteq A$ and thus also $f(i) \in A - dom\phi_i = A$. We let that be the first element of our infinite set, $x_0 = i$. Now assume that we have generated elements $x_0, x_1, \ldots, x_n$ of our new set, in that order, and let us generate element x_{n+1}. There is a partial recursive function with domain exactly equal to $\{x_0, x_1, \ldots, x_n\}$, call it ϕ_{j_n}—how to generate this function is a problem we shall tackled shortly. Then we have $dom\phi_{j_n} \subseteq A$ and thus $f(j_n) \in A - dom\phi_{j_n}$; in other words, $f(j_n)$ is an element of A not in the set $\{x_0, x_1, \ldots, x_n\}$, and so is exactly the new element we wanted, $x_{n+1} = f(j_n)$. Since we have a recursive procedure to enumerate an infinite subset of A, the subset is r.e., as desired.

That the function ϕ_{j_n} can be constructed may at first appear difficult, but it, too, can be constructed recursively. We have the first such function, the totally undefined function. Assume that we have already produced the function $\phi_{j_{n-1}}$ with domain $\{x_0, x_1, \ldots, x_{n-1}\}$; we can define ϕ_{j_n} as follows:

$$phi_{j_n}(y) = \begin{cases} 1 & y = f(j_{n-1}) \\ \phi_{j_{n-1}}(y) & \text{otherwise} \end{cases}$$

and a standard s-1-1 construction produces the index j_n from the index j_{n-1}.