The Theory of Computation

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Solution Manual

Chapter 5, Problem 5.29

1. To show that a set and its cylindrification belong to the same m-degree, we need to demonstrate many-one reductions from each to the other. Reducing s to $S \times \mathbb{N}$ is trivial: we can simply map each value x to the pair $(x, 0)$. Clearly, x belongs to S if and only if $(x, 0)$ belongs to $S \times \mathbb{N}$. The converse is equally easy: given a pair (x, y), we map it to the value x; again, it is clear that (x, y) belongs to $S \times \mathbb{N}$ if and only if x belongs to S.
2. We know that simple sets are not creative. This part of the problem then asks us to show that this property is preserved through cylindrification. (But simplicity may not be: we are not asked to show that $S \times \mathbb{N}$ is simple, only that it is not creative—in fact, it is not simple.) Let S be a simple set and $S \times \mathbb{N}$ its cylindrification. Since S is r.e., so is $S \times \mathbb{N}$ (any enumerator for S can be turned into an enumerator for $S \times \mathbb{N}$ through simple dovetailing); thus we need to show that the complement of $S \times \mathbb{N}$, which is just $\overline{S} \times \mathbb{N}$, is not productive.
   Assume then that $\overline{S} \times \mathbb{N}$ is productive with productive function f. We want to show that we can use f to construct a productive function for $\overline{S}$, which would contradict the simplicity of S and prove our result. Let ϕ be some partial recursive function with $dom\phi \subseteq \overline{S}$; then the function $\theta(\langle x, y\rangle) = \phi(x)$ clearly obeys $dom\theta \subseteq S \times \mathbb{N}$; moreover, we have $dom\theta = dom\phi \times \mathbb{N}$. If we write $\phi_j(\langle x, y\rangle) = \theta(\langle x, y\rangle)$, then we have $f(i) \in (\overline{S} \times \mathbb{N}) - dom\theta$, that is, $f(i) \in (\overline{S} \times \mathbb{N}) - (dom\phi \times \mathbb{N})$, that is $f(i) \in (\overline{S} - dom\phi) \times \mathbb{N}$. But $f(i)$ is some pair $\langle x, y\rangle$, so that $\Pi_1(x) \in \overline{S} - dom\phi$. In other words, $\Pi_1 \cdot f$ is a productive function for $\overline{S}$, the desired contradiction.