The Theory of Computation

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Solution Manual
Chapter 5, Problem 5.30

This is a rather open-ended exercise. To circumscribe it, let us consider the various results of Section 5.7.

The first result (Proposition 5.2) was that there is a unique m-degree that contains exactly the nontrivial recursive sets. As mentioned in the problem statement, such is no longer the case with 1-degrees: for instance, in order for two finite sets to reduce to each other through one-reductions, they must be of the same size. Let us then prove what is given in the problem statement: namely that the recursive sets are partitioned into the following equivalent classes: finite sets (one class for each size), infinite sets with finite complements (again, one class for each size of complement), and infinite sets with infinite complements (a single class). In order to reduce finite set A to finite set B with an injective map, we must have $|A| \leq |B|$; thus in order for the two sets to be in the same 1-degree, we must have $|A| = |B|$. Now, if two finite sets have the same size, we claim that we can always reduce one to the other: basically, we can order the elements of the universe and define a map that is essentially an identity, except on the elements that belong to one of the two sets; we map the elements of A to those of B following the ordering and map the elements of $B - A$ to elements of $A - B$, again according to the ordering. Thus all finite sets of the same size form a 1-degree. Essentially the same argument shows that infinite sets with finite complements are partitioned the same way, according to the size of their complement (the mapping is one-to-one on the universe and thus in particular on the complements, with exactly the consequences described above). Finally, we claim that infinite recursive sets with infinite complements form a single equivalence class: we can enumerate each set and each complement and define the mapping through the enumeration—map the ith element of the first set to the ith element of the second and the j element of the complement of the first set to the jth element of the complement of the second. In terms of lattice of 1-degrees, we see that we can one-reduce a small finite set to a larger finite set, but not vice-versa; similarly, we can reduce an infinite set with a small finite complement to an infinite set with a larger finite complement, but not vice-versa; finally, we can reduce either type of set to an infinite set with infinite complement. Thus the lattice has a single element at the top, the 1-degree of infinite recursive set with infinite complement and has two infinite "columns" underneath, one formed by 1-degrees of finite sets, with size 0 at the bottom and increasing by one at each level, and one formed by 1-degrees of infinite sets with finite complements, with complement size 0 at the bottom and complement size increasing by one at each level. Comparing with Figure 5.5, we see that both have $\{\emptyset\}$ and $\{\mathbb{N}\}$ at the bottom, but that many-one reductions merge all non-zero sizes (finite or infinite).

Our next result is Proposition 5.3, which states that an m-degree that contains an r.e. set contains only r.e. sets. That result is completely unaffected by the nature of the reduction, since it uses only composition of functions.

Our definition of many-one completeness is changed into the obvious definition of one-completeness. Theorem 5.8 remains true with the change: K is one-complete for the class of r.e. sets: this is due to an earlier observation that the s-m-n function is in fact an injective function (our Theorem 5.1 constructs it mostly through pairing, which is injective). Proposition 5.4, on the other hand, is clearly false for on-completeness, in view of the lattice we just discussed; it needs to be modified to "any infinite recursive set with infinite complement is one-complete for the class of recursive sets." *Ditto* with the comment just below: only infinite recursive sets with infinite complements one-reduce to their complement.

The discussion of productive, creative, and simple sets is not changed, since their definitions do not involve a specific type of reduction. Thus Theorem 5.10 remains unchanged. Somewhat surprisingly, so does Theorem 5.9: we can in fact state that an r.e. set if many-one complete for the r.e. sets if and only if it is one-complete for them and if and only if it is creative. The "only if" part of the proof is clearly unaffected: creativity does not depend on the type of reduction used. The "if" part is the interesting one, since we need to show that the induced reduction $f \cdot e$ is injective. Once again, though, this follows rather directly from the fact that the s-m-n function is injective. The least-upper-bound argument of Theorem 5.11 clearly remains unaffected, since the mappings used are injective.

Finally, we want to consider cylindrifications. We proved in Exercise 5.29 that a set and its cylindrification belong to the same m-degree, but only one of the two mappings used in that proof was injective. Indeed, it is pretty clear that a finite set and its cylindrification cannot belong to the same 1-degree, since we cannot one-reduce the infinitely large cylindrification to the finite set. The cylindrification of any nontrivial recursive set is an infinite recursive set with infinite complement and thus sits at the top of the lattice. Thus exceptions can only arise at the top or at the very bottom of the lattice. The cylindrification of the empty set is still empty and the cylindrification of the universe remains the universe; in those cases, the set and its cylindrification do belong to the same 1-degree at the very bottom of the lattice. And the cylindrification of an infinite recursive set with infinite complement is another infinite recursive set with infinite complement, so that these sets and their cylindrifications do belong to the same 1-degree, at the very top of the lattice.