The Theory of Computation

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Solution Manual
Chapter 5, Problem 5.8

The reason is that the definition of the new "diagonal" function, $g(n) = f_n(n) + 1$, is not a primitive recursive definition—it cannot always be defined with just substitution and primitive recursion. We saw a detailed example when looking at Ackerman's function, which uses much the same definitional scheme, namely, "use the nth function on the nth argument," and showed then that the function thus defined was not primitive recursive.

The key to a contradiction proof by diagonalization always resides in constructing a new object that is not in the set (by virtue of its construction), but which in fact ought to be in the set if the claimed negated result were to be true, yielding the desired result through contradiction. The proposed construction does yield a new total function, but we have no reason to believe it to be in the set of primitive recursive functions; on the other hand, the very same argument applied to all computable total functions works, because we have very few restrictions left when defining arbitrary (partial or not) recursive functions and thus the new function g should now be in the set if computable total functions.