The Theory of Computation
Bernard M.E. Moret
Addison-Wesley-Longman, 1998

Solution Manual
Chapter 5, Additional Problem 5.1

1. Denote the least common multiple of x and y by $\text{lcm}(x, y)$; since the lcm of x and y cannot exceed xy, we simply write

$$\text{lcm}(x, y) = \min z \leq \text{mult}(x, y)\ [x|z \text{ and } y|z]$$

2. Denote the greatest common divisor of x and y by $\text{gcd}(x, y)$; we simply write

$$\text{gcd}(x, y) = \max z \leq x\ [z|x \text{ and } z|y]$$

3. Let the number of divisors of x be $d(x)$ and the number of prime divisors of x be $pd(x)$. In both this problem and the next, we need a primitive recursive construct analogous to a bounded search that returns the number of distinct $z \leq x$ that meet a certain predicate. We define such a construct below, denoting it $\text{count}(z \leq x)[P(z)]$. Then we have

$$d(x) = \text{count}(z \leq x)[z|x]$$

and

$$pd(x) = \text{count}(z \leq x)[(z|x) \text{ and is\_prime}(z)]$$

To define the count schema, we use primitive recursion as follows (assuming that P is a primitive recursive predicate):

$$\begin{cases} \text{count}(z \leq 0)[P(x)] = P(x) \\ \text{count}(z \leq x+1)[P(x)] = \text{add}(P(x+1), \text{count}(z \leq x)[P(x)] \end{cases}$$

4. Euler's function is generally written $\phi(x)$. Using the count construct, we write

$$\phi(x) = \text{count}(z \leq x)[gcd(z, x) = 1]$$

5. Denote the leading nonzero bit in the unsigned binary representation of x by $\text{msb}(x)$. We write

$$\begin{cases} \text{msb}(1) = 0 \\ \text{msb}(x+1) = \begin{cases} \text{Succ}(\text{msb}(x)) & \text{is\_power\_of\_2}(x+1) \\ \text{msb}(x) & \text{otherwise} \end{cases} \end{cases}$$

and define

$$\begin{cases} \text{is\_power\_of\_2}(1) = 1 \\ \text{is\_power\_of\_2}(x+1) = \left[x + 1 = \text{mult}(2, \max z \leq x\ [\text{is\_power\_of\_2}(z)])\right] \end{cases}$$