The Theory of Computation

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Solution Manual
Chapter 5, Additional Problem 5.7

Let us give a reduction-based proof rather than a direct proof. We prove that productivity is preserved under many-one reductions; then we reuse the construction from our proof of Rice's theorem to show that $\overline{K}$, a known productive set, reduces to the complement of any r.e. set obeying the hypothesis of Rice's theorem, thereby proving our result.

Let A be a productive set with productive function f and let h be a many-one reduction from A to B. Then it is easily seen that $h \cdot f$ is a productive function for B, since we have $x \in A \Leftrightarrow h(x) \in B$.

Now consider the construction of Rice's theorem. If the family of functions $\mathscr{C}$ contains the totally undefined function ϕ_u, then we claim that $\overline{K}$ reduces to our set $P_{\mathscr{C}} = \{x \mid \phi_x \in \mathscr{C}\}$, which is then productive and thus not r.e. It follows that our r.e. family of functions cannot contain the totally undefined function, so that its complement does contain the totally undefined function, so that, by the same reasoning, $P_{\overline{\mathscr{C}}}$ is productive, so that $P_{\mathscr{C}}$ is creative, as desired.

To verify the key claim, let ψ be some function not in $\mathscr{C}$ and let us define the function

$$\theta(x, y) = \psi(y) + \text{Zero}(\phi_{\text{univ}}(x, x))$$

and use an s-1-1 construction to obtain a function f such that $\phi_{f(x)}(y) = \theta(x, y)$. Now, if x belongs to $\overline{K}$, then $\theta(x, y)$ is totally undefined, so that $f(x)$ belongs to $P_{\mathscr{C}}$; conversely, if x belongs to K, then $\theta(x, y)$ is the function $\psi(y)$, so that $f(x)$ does not belong to $P_{\mathscr{C}}$. Hence we have reduced $\overline{K}$ to any nontrivial set of indices for a family of functions that includes the totally undefined function, showing that such a set of indices is productive.