The Theory of Computation

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Addison-Wesley-Longman, 1998

Solution Manual
Chapter 6, Problem 6.16

That's a very open-ended question. Rather than try to resolve it, we simply give a few examples of what can be done. The basic question asked here is how much of a nondeterministic hierarchy for space can be built without having a hierarchy theorem for nondeterministic space? Proving a nondeterministic version of our hierarchy theorems is quite challenging, so a simpler alternative, at least for space, might be to use existing tools to build the hierarchy indirectly. Our tools are

1. the hierarchy theorem for deterministic space
2. the basic relationship $\text{DSPACE}(f) \subseteq \text{NSPACE}(f)$
3. Savitch's relationship $\text{NSPACE}(f) \subseteq \text{DSPACE}(f^2)$
4. translational lemmata, including Lemma 6.3: $\text{SPACE}(g_1) \subseteq \text{SPACE}(g_2) \Rightarrow \text{SPACE}(g_1 \cdot f) \subseteq \text{SPACE}(g_2 \cdot f)$, for f at least as large as n. but also what we called out downward separation results: if nondeterminism no longer pays at some level in the hierarchy, it cannot pay any higher up in the hierarchy.

The key is Savitch's theorem, since it gives us a rather narrow bracket (a tight "upper bound") for the nondeterministic class to sit in—the same program of construction for nondeterministic time would fail completely because the upper bound is much too high.

For instance, the hierarchy theorem for space tells us that quadratic space is a proper subset of cubic space; it follows that nondeterministic linear space is a proper subset of nondeterministic cubic space, because we can write

$$NSPACE(n) \subseteq \text{DSPACE}(n^2) \subset \text{DSPACE}(n^3) \subseteq \text{NSPACE}(n^3)$$

(Of course, we can tighten this relationship to state that nondeterministic linear space is a proper subset of any super-quadratic nondeterministic space class, since that is all we need to invoke the hierarchy theorem.)

The problem with this approach is that the gap imposed by Savitch's theorem seems unbridgeable; for instance, if we want to consider the class $\text{NSPACE}(n^3)$, the best this strategy will get is is that it is a proper subset of any class $\text{NSPACE}(n^{6+\varepsilon})$. Using translational lemmata helps us overcome this part of the problem. We can prove, for instance, that $\text{NSPACE}(n^3)$ is a proper subset of $\text{NSPACE}(n^4)$ by contradiction: if we have $\text{NSPACE}(n^3) = \text{NSPACE}(n^4)$, then we can translate this equality by applying $g(x) = x^{4/3}$ to both resource bounds, obtaining $\text{NSPACE}(n^4) = \text{NSPACE}(n^{16/3})$, which allows us to conclude $\text{NSPACE}(n^3) = \text{NSPACE}(n^{16/3})$; then we can again translate with the same g to obtain $\text{NSPACE}(n^{16/3}) = \text{NSPACE}(n^{64/9})$, and thus conclude $\text{NSPACE}(n^3) = \text{NSPACE}(n^{64/9})$. But this last equality contradicts our earlier result that $\text{NSPACE}(n^3)$ is a proper subset of of any class $\text{NSPACE}(n^{6+\varepsilon})$, so that we must in fact have had $\text{NSPACE}(n^3) \subset \text{NSPACE}(n^4)$, as desired.

The technique just demonstrated can be applied over and over to amplify difference in resource functions for space until we contradict a firm gap obtained by a combination of the hierarchy theorem for deterministic space and Savitch's theorem. The result is a very detailed hierarchy for nondeterministic space.