The Theory of Computation
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Solution Manual
Chapter 6, Problem 6.7

What we need to prove is that, if some problem A is in PolyL and some problem B many-one reduces to A in logarithmic space, then B is also in PolyL. Since A is in PolyL, it is in L^i for some $i \in \mathbb{N}$. We need to visualize the reduction process on our off-line Turing machines. What happens is that the reduction takes an instance x of B and, using logarithmic space—that is, $O(\log |x|)$—on the scratch tape, produces an instance $f(x)$ of A on the output tape, which is then used as input tape by the machine for A. That machine, in turn, uses L^i space—that is $O(\log^i |f(x)|)$—on its scratch tape to decide the instance. If we ignore the space used to communicate $f(x)$, what we need to show is that $\log^i |f(x)|$ is polylogarithmic in $|x|$, the size of the input. Note that the reduction process through f take $O(\log |x|)$ space and thus cannot take more than $O(c^{\log |x|})$ time; in particular, the output it produces, $f(x)$, cannot be longer than the number of steps used by the reduction and thus has length $O(c^{\log |x|})$. Thus the space used by the decision procedure for A on instance $f(x)$ is $O(\log^i |f(x)|) = O(\log^i c^{\log |x|}) = O(\log^i |x|)$, as desired.

The issue of the space taken by the output of the reduction, before that is used up by the decision procedure for the target set, is taken up in Exercise 6.11.