The Theory of Computation

Bernard M.E. Moret

Addison-Wesley-Longman, 1998

Solution Manual
Chapter 8, Problem 8.12

An instance of the *Minimum Complete Research Program* problem is given by a set X of "open problems," a partial order on X, and a positive integer $B \leq |X|$. The question is whether there exists a subset $X' \subseteq X$, $|X'| \leq B$, and a "complexity classification" function $c\colon X' \to \{\text{easy, hard}\}$ that would resolve all the open problems, i.e., such that c can be extended to a well-defined total function on X by using the partial order:

- $x < y$ and $c(x) = \text{hard} \Rightarrow c(y) = \text{hard}$
- $x < y$ and $c(y) = \text{easy} \Rightarrow c(x) = \text{easy}$

We prove this problem NP-complete (it is obviously in NP) by transforming *Vertex Cover* to it. Given an instance of *VC*, a graph $G = (V, E)$ and a positive integer $K \leq |V|$, we produce an instance of *MCRP* as follows. Let $l = |V| + K$. For each vertex $v \in V$, we create $l + 2$ vertices, $v_1, \ldots, v_l, v', v''$, and $l + 1$ arcs, (v_i, v') for $1 \leq i \leq l$, and (v', v''). For each edge, $(v, w) \in E$, we create l vertices, $e_1, \ldots, e_l$ and $2l$ arcs, (v'', e_i) and (w'', e_i), for $1 \leq i \leq l$. The bound on the size of the research program is set to l. This transformation is clearly feasible in polynomial time.

Now, if we have a vertex cover, $U \subseteq V$, of size K or less, we can obtain a complete research program of size l or less by classifying each node v' ($v \in U$) hard, all nodes v'' ($v \notin U$) easy, and all nodes w' ($w \notin U$) easy. (It follows that all remaining w'' nodes and all e_i nodes get classified as easy and all v_i nodes as hard.) Conversely, a complete research program of size l or less must yield a vertex cover of size K or less. By construction, the e_i nodes can only be classified as easy and the v_i nodes as hard; moreover, none of these nodes can be part of the l nodes. If the research program contains only one of v' and v'' for some v, then it contains v'' with classification easy. The rest follows.

A somewhat easier transformation from *VC* proceeds as follows: for each vertex v, set up three problems v, v', v'' with $v' < v$ and $v' < v''$; for each edge e, set up one problem e, easier than both its endpoints; and set the bound to $|V| + K$. The correctness of the transformation is easily verified.