The Theory of Computation
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Solution Manual
Chapter 8, Problem 8.2

We prove that *Planar NAE3SAT* is in P by reduction to a known problem in P, namely *Planar MaxCut*. Our reduction uses only local replacement and carefully expands the single variable nodes of *Planar NAE3SAT* into chains of nodes for *MaxCut* so as to produce only planar instances of *MaxCut*. Orlova and Dorfman ("Finding the maximum cut in a graph," *Eng. Cybernetics* **10** (1972), 502–506) showed that such instances can be solved in polynomial time by a matching process carried on the dual of the graph. Thus *Planar NAE3SAT* is in P.

An instance of (Nonpolar) *Planar NAE3SAT* is defined as an instance of *NAE3SAT*, except that the graph produced from it according to the following rules must be planar. The graph has one vertex for each variable of the instance and one vertex for each clause of the instance; all variable vertices are connected in a simple cycle and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause. Note that positive and negative literals are treated exactly alike. (Simple) *MaxCut* is defined as follows: given an undirected graph and given a bound (a positive integer), can the vertices of the graph be partitioned into two subsets such that the number of edges with endpoints in both subsets is no smaller than the given bound?

Our transformation is quite elementary. Given an instance of *Planar NAE3SAT* with k clauses, we transform it in polynomial time into an instance of *MaxCut* with $9k$ vertices and no more than $6k$ edges as follows. For each variable x forming a total of n_x literals in the k clauses, we set up a simple cycle of $2n_x$ vertices (and $2n_x$ edges); alternating vertices represent complemented and uncomplemented literals.[1] For each clause, we set up a triangle, where each vertex of the triangle is connected by an edge to the (complement of the) corresponding "literal vertex." Finally, we set the

[1] Note that, if $n_x = 1$, this would produce a double edge between two vertices; in this case only, we set up a single edge between the two vertices and lower the required number of cut edges by one. Since this is an elementary adjustment we shall assume in the following that $n_x \geq 2$ rather than qualify every statement to take this special case into account.

minimum number of edges to be cut to $11k$ This transformation is clearly feasible in polynomial time.

Now, given a satisfying truth assignment, we put all vertices corresponding to true literals on one side of the partition and all others on the other side. Since the truth assignment is valid, each edge between a literal and its complement is cut, thereby contributing a total (for all variables) of $6k$ (i.e., twice the total number of literals) to the cut sum. Since the truth assignment is a solution, each triangle is cut (not all three vertices may be on the same side, as this would correspond to a clause with three false or three true literals), thereby contributing a total of $5k$ to the cut sum. Hence we have a solution to *MaxCut*. Conversely, observe that $11k$ is the maximum attainable cut sum: we cannot do better than cut each clause triangle and each segment between complementary literals. Moreover, the cut sum of $11k$ can only be reached by cutting all triangles and segments. (If all three vertices of a clause triangle are placed on the same side of the partition, at most 4 of the 6 edges associated with the clause can be cut.) Hence a solution to *MaxCut* yields a solution to *NAE3SAT*: cutting the segments ensures a valid truth assignment and cutting each triangle ensures a satisfying truth assignment.

Only one point remains: to verify that the instance produced is indeed planar. No crossing can arise from the replacement of the single clause vertices of *Planar NAE3SAT* the triangles of *MaxCut*, as the vertices of the triangles are assigned arbitrarily. The replacement of the single variable vertices of *Planar NAE3SAT* by the simple cycles of *MaxCut* likewise cannot produce non-planar configurations, as the choice of vertex pair for some literal $\tilde{x}$ from among the n_x pairs present in the circle is also arbitrary. The correctness of this transformation completes our proof.

It is interesting to note that rather more transformations from the *3SAT* family of problems to graph problems produce a single edge per variable (with the endpoints intended to represent the two truth value assignments) than produce a single vertex or arbitrarily labelled structure. Thus a potentially more useful definition of *Planar 3SAT* would constrain instances according to a different graph construction, where each variable is in fact represented by a single edge and where the single vertex clause is connected to the proper (i.e., complemented or not) endpoint of the variable's edge—this is the polar version of *Planar 3SAT*. Simplifying the transformation given above to one where a single edge (rather than a simple cycle) is produced for each variable—thus yielding the simple transformation given in the text—shows that *Planar NAE3SAT* remains in P under this new definition.