The Theory of Computation

Bernard M.E. Moret

Addison-Wesley-Longman, 1998

Solution Manual
Chapter 8, Problem 8.25

Clearly, if the optimal solution has no more than k objects in it, our algorithm will discover the optimal solution. Assume then that the optimal solution has over $n > k$ objects and consider the subset of k objects formed by the k most valuable objects in the optimal solution. This subset is one of the $\left(\binom{n}{k}\right)$ subsets examined by our algorithm; we claim that its greedy completion (through ordering by value density) yields a solution of value no less than $\frac{k}{k+1}$ times the optimal.

Denote the value of our subset of k objects by V and its size by S; denote the value of the optimal solution by $\hat{V}$. We know that the $n-k$ other elements in the optimal solution (each of value not exceeding that of the least valuable element in our subset) take up no more than $B-S$ space, so that at least one of these objects must have size at most $\frac{B-S}{n-k}$. The greedy completion attempts to pack the remaining $B-S$ space with the items that give the best value for their size; it will differ from the optimal solution only if it finds remaining objects of higher value density than the $n-k$ present in the optimal solution. In the worst case, one such object is found and its inclusion precludes that of any other object; in view of our previous comment, that means that less than $\frac{B-S}{n-k}$ space is left—or, equivalently, that the included object has size at least $(B-S)\frac{n-k-1}{n-k}$. Since its value density is as high as any in the $n-k$ other objects in the optimal solution, our greedy completion must have a value at least $\frac{n-k-1}{n-k}$ that of the optimal completion (which is, in our notation, $\hat{V}-V$). Overall, then, our solution (the greedy completion of the best k elements of the optimal solution) has value at least $V + \frac{n-k-1}{n-k}(\hat{V}-V)$, which we can simplify to $\frac{V+(n-k-1)\hat{V}}{n-k}$ Its ratio to the optimal solution is therefore at least $\frac{n-k-1}{n-k} + \frac{V}{(n-k)\hat{V}}$; since V represents the k best items of n items worth $\hat{V}$ in all, we must have $V \geq \frac{k}{n}\hat{V}$, so that our ratio is at least $1 - \frac{1-\frac{k}{n}}{n-k} = 1 - \frac{1}{n}$. The value n is not known in advance, but we have assumed $n > k$, so that we can write that the ratio is at least $1 - \frac{1}{k}$, as desired.