The Theory of Computation
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Solution Manual
Chapter 8, Additional Problem 8.3

We shall show that, given an ε-approximation for *DCC*, we could decide the *HC* problem. Thus, suppose that we have an approximation algorithm for *DCC* such that the size of the optimal cover, $S*$, and that of the approximate solution, $\hat{S}$, satisfy $\hat{S}/S* \leq \varepsilon$. Given a graph, $G = (V, E)$ for the *HC* problem, we construct a new graph, *G*', on which we run our ε-approximation; depending on the size of the *DCC* obtained, we answer yes or no to the *HC* problem. We construct *G*' as follows.

Pick a vertex, $v \in V$, and let U_v be the set of vertices adjacent to v; call n_v its cardinality, i.e., $U_v = \{u_1, \ldots, u_{n_v}\}$. *G*' will consist of n_v unconnected subgraphs, $G_1, \ldots, G_{n_v}$. Let k be the smallest integer larger than ε, $k = \lfloor \varepsilon \rfloor + 2$. Then each G_i is composed of k isomorphic copies of the original G, together with a specific interconnection pattern, which links all k copies in a cycle. Specifically, G_i has extra edges $\{u_{i,j}, v_{j+1}\}$, where $u_{i,j}$ is the i-th vertex of U_v in the j-th isomorphic copy of G and v_{j+1} is the picked vertex v in the $j + 1$-th isomorphic copy of G, and the extra edge $\{v_k, u_{i,1}\}$.

Now, since the k isomorphic copies are connected in a simple cycle which just replaces the original $\{u_i, v\}$ edge of G, no G_i has a Hamiltonian circuit unless G has one; moreover, if G does not have an *HC*, then the minimal *DCC* for each G_i has at least $k + 1$ cycles (since no two added edges can belong to distinct cycles). Conversely, if G has an *HC*, then this circuit must include one of the $\{u_i, v\}$ edges; then G_i has an *HC* (because each copy has a Hamiltonian path from u_i to v and we can use the additional edges to complete the circuit).

Thus, we now run our *DCC* approximation algorithm on each G_i in turn; if the size of the approximate *DCC* for any of them is no larger than k, we answer yes to the *HC* problem; if the size of the approximate *DCC* is larger than k for all of them, we answer no to the *HC* problem.

Note that there are as many possible Turing reductions for this problem as there are people devising such; the above is only one example—deliberately a bit different from our standard approach to such problems.